JOURNAL OF APPROXIMATION THEORY 29, 212-217 (1980)

Restricted Approximation by Strongly Sign-Regular Kernels: The Finite Bang–Bang Principle

KLAUS GLASHOFF

Institut für Angewandte Mathematik, Universität Hamburg, D-2000 Hamburg 13, West Germany

Communicated by Samuel Karlin

Received November 27, 1978

DEDICATED TO THE MEMORY OF P. TURÁN

1. INTRODUCTION AND DESCRIPTION OF THE RESULT

Let $K(s, t)$ be a real valued function defined on $S \times T$ where $S = (a, b)$ and $T = (c, d)$ are bounded intervals of the real line. More exact requirements on $K(s, t)$ will be specified later. In this paper we consider the problem of best uniform approximation of a continuous function $z(s)$ on S by functions of the form

$$y(s) = \int_T K(s, t)\, u(t)\, dt, \tag{1.1}$$

where $u(t)$ is subject to the inequality constraint

$$|u(t)| \leqslant 1 \qquad \text{for all } t \text{ in } T. \tag{1.2}$$

Thus we wish to minimize

$$\max_{s \in S} \left| z(s) - \int_T K(s, t)\, u(t)\, dt \right| \tag{1.3}$$

under the restriction (1.2).

Problems of this type arise in control theory where (1.1) defines a control operator with $u(t)$ as control function and $z(s)$ as the "desired state" (cf. [2]).

Our main result is the following. We show that under suitable assumptions on the kernel $K(s, t)$ there is a unique solution $\bar{u}$ of the restricted approximation problem such that $\bar{u}$ is *finitely bang–bang*:

$$\bar{u}(t) = \epsilon(-1)^r \qquad \text{for} \quad t \in [t_{r-1}, t_r); \quad r = 1, \ldots, n+1, \tag{1.4}$$

0021-9045/80/070212-06$02.00/0

for some $\epsilon = +1$ or -1, some $n \geqslant 1$ and some partition

$$c = t_0 < t_1 < \cdots < t_n < t_{n+1} = d$$

of the interval T.

In order to obtain such a result we require that $K(s, t)$ is *strongly sign-regular* (SSR), i.e., that there exists a sequence of numbers ϵ_m either $+1$ or -1 such that

$$\epsilon_m K \binom{s_1, \dots, s_m}{t_1, \dots, t_m} = \epsilon_m \det\{K(s_i, t_k)\} > 0 \tag{1.5}$$

for all $a < s_1 \cdots s_m < b$, $c < t_1 < \cdots < t_m < d$. Kernels of this type have well-known *variation diminishing* properties which we use in the proof of our result.

It is possible to apply our theorem to a broad class of control problems, for example to the boundary control of the heat equation. For this class of problems we can derive the finite bang–bang theorem of Karafiat [4] in a much simpler way. A detailled discussion of this result and related applications will be given elsewhere.

2. Characterization of Optimal Solutions

In order to give a simple presentation of the results we assume that $K(s, t)$ is continuous on the closure of $S \times T$. This ensures that (1.1) defines a continuous linear operator from the space $L_\infty(T)$ of bounded measurable functions on T into the space $C(\bar{S})$ of continuous functions on $\bar{S} = [a, b]$. Let us denote this operator by the symbol K, too:

$$(Ku)(s) = \int_T K(s, t)\, u(t)\, dt, \qquad s \in S. \tag{2.1}$$

We introduce the following notation:

$$B = \{u \in L_\infty(T) / | u(t)| \leqslant 1 \text{ a.e. on } T\},$$

where we always understand that T is equipped with ordinary Lebesgue measure.

Now our problem certainly can be formulated as follows:

$$\underset{y \in K(B)}{\text{Minimize}} \| z - y \|_\infty , \tag{2.2}$$

where z is a fixed given element in $C(\bar{S})$ and $\| \cdot \|_\infty$ is the uniform norm:

$$\| x \|_\infty = \max_{s \in \bar{S}} | x(s)| \qquad \text{for} \quad x \in C(\bar{S}).$$

The following existence result can be proved in a standard way, and we therefore omit the proof (cf. [2]).

LEMMA 1. *There is a $\bar{y}$ in $K(B)$ such that*

$$\| z - \bar{y} \|_\infty \leqslant \| z - y \|_\infty \qquad \text{for all } y \text{ in } K(B).$$

The topological dual of $C(\bar{S})$ is well known to be isomorphic to the space of measures $d\alpha$ induced by functions α on $\bar{S}$ of bounded variation. We specialize the standard characterization theorem of linear approximation theory (cf. Holmes [3, Sect. 22]) to our case and obtain:

THEOREM 1. *$\bar{y}$ is solution of (2.2) if and only if there is a measure $d\alpha$ such that*

$$\int_S \{z(s) - \bar{y}(s)\}\, d\alpha(s) = \| z - \bar{y} \|_\infty , \tag{2.3}$$

$$\int_S \bar{y}(s)\, d\alpha(s) = \max_{y \in K(B)} \int_S y(s)\, d\alpha(s). \tag{2.4}$$

Let us draw an important conclusion from this theorem. If $\bar{y}$ is a solution of our problem, let $\bar{u} \in B$ be such that

$$\bar{y} = K(\bar{u}).$$

We define

$$\lambda(t) = \int_S K(s, t)\, d\alpha(s), \qquad t \in T, \tag{2.5}$$

where $d\alpha$ is the measure appearing in Theorem 1. Using this notation, we write (2.4) in the form

$$\int_T \bar{u}(t)\, \lambda(t)\, dt = \max_{u \in B} \int_T u(t)\, \lambda(t)\, dt, \tag{2.6}$$

which we call the *maximum principle.* It immediately implies

$$\bar{u}(t) = \operatorname{sgn} \lambda(t) \qquad \text{for } \lambda(t) \neq 0. \tag{2.7}$$

Now it is clear that we are going to prove our main result by showing that $\lambda(t)$ has only a finite number of zeroes on T. This is the subject of the next section.

3. The Finite Bang–Bang Principle

Let $d\alpha = d\alpha^+ - d\alpha^-$ be the decomposition of the measure $d\alpha$ into the difference of two nonnegative measures such that supp $d\alpha^+ \cap$ supp $d\alpha^- = \varnothing$. Here supp denotes the support of a measure.

It is well known (cf. Holmes [3, p. 81, Exercise 33]), that (2.3) implies

$$\begin{aligned} \operatorname{supp} d\alpha^+ \subset P^+ &= \{s \in \bar{S}/z(s) - \bar{y}(s) = \| z - \bar{y} \|_\infty\} \\ \operatorname{supp} d\alpha^- \subset P^- &= \{s \in \bar{S}/z(s) - \bar{y}(s) = -\| z - \bar{y} \|_\infty\}. \end{aligned} \tag{3.1}$$

The following almost trivial lemma gives us the main tool for the proof of our theorem.

Lemma 2. *Let $z \neq \bar{y}$. Then there is a partition*

$$a = s_0 < s_1 < \cdots < s_m < s_{m+1} = b$$

of S such that, for each $S_i = (s_{i-1}, s_i)$, the intersection of S_i with one of the sets P^+ or P^- is empty.

We do not want to waste space for a proof of this simple lemma which is just a formal statement of the fact that a continuous function on a compact interval cannot oscillate between its absolute extrema infinitely often unless it is constant.

Corollary 1. *Let $\| z - \bar{y} \|_\infty > 0$ where $\bar{y}$ is a solution of the approximation problem. If $d\alpha$ is a measure satisfying* (2.3), *then there exists a partition of the interval S into finitely many subintervals $S_1, \ldots, S_{m+1}$ such that the sign of the measure induced on the subintervals by $d\alpha$* alternates; *i.e., for any $y \in C(\bar{S})$*

$$\int_{\bar{S}} y(s)\, d\alpha(s) = \epsilon \sum_{i=1}^{m+1} (-1)^i \int_{\bar{S}_i} y(s) \mid d\alpha(s) \mid,$$

$\epsilon = +1$ or -1, where it can be assumed that $\int_{\bar{S}_i} d\alpha(s) \neq 0$, $i = 1, \ldots, m+1$.

The preceding corollary follows immediately from the fact that $d\alpha \neq 0$ (because of (2.3) and $\| z - \bar{y} \|_\infty > 0$), Eqs. (3.1) and Lemma 1.

If a measure $d\alpha$ has the alternation property described above, then we write

$$S^-(d\alpha) = m$$

and say that $d\alpha$ has *exactly m sign changes* on S.

Our aim is now to obtain a bound on the number of zeroes of the function $\lambda(t)$ defined by (2.5).

DEFINITION 1. (cf. [5, p. 230]). Let $x = (x_1, \ldots, x_n)$ be a vector of real numbers. We denote by $S^+(x)$ the maximum number of sign changes possible in the vector x by allowing each zero to be replaced by $+1$ or -1. The number $S^+(f)$ of sign changes of a real valued function $f(t)$ defined on an ordered subset T of the real line, is $\sup S^+(f(t_1), \ldots, f(t_n))$, where the supremum is extended over all sets $t_1 < t_2 < \cdots < t_n$ ($t_i \in T$; n arbitrary but finite).

It is clear by definition of $S^+(f)$ that $S^+(f) < \infty$ implies that f has only finitely many zeroes.

THEOREM 2. *Let $K(s, t)$ be SSR on $S \times T$ (cf. (1.5)), and assume that*

$$S^-(d\alpha) = m.$$

Then

$$S^+(\lambda) \leqslant m,$$

where

$$\lambda(t) = \int_{\bar{S}} K(s, t)\, d\alpha(s), \qquad t \in T.$$

We do not want to prove this theorem in detail because it seems to belong to the well-known facts on SSR kernels. For sake of completeness we just give the main idea of the proof, modelled after the one given in [5, p. 234], for a related problem.

We define

$$\phi(t, i) = \int_{\bar{S}_i} K(s, t)\, |\, d\alpha(s)| \qquad (i = 1, \ldots, m + 1).$$

Then, for any partition $t_1 < t_2 < \cdots < t_n$, $t_k \in T$,

$$\lambda(t_k) = \{\operatorname{sgn} \lambda(S_1)\} \sum_{i=1}^{m+1} (-1)^{i+1}\, \phi(t_k, i).$$

The composition formula for determinants [5, p. 98] shows that the matrix $\{\phi(t_k, i)\}$ is SSR. The desired result follows immediately by a theorem on the variation diminishing properties of strongly sign-regular matrices (Gantmacher and Krein [1, p. 285]).

Collecting our results, we come to the final formulation of our theorem announced in the introduction.

THEOREM 3. *Let $\min_{u \in B} \| z - Ku \|_\infty > 0$. If $K(s, t)$ is SSR, then any solution $\bar{u}$ is finitely bang–bang.*

COROLLARY 2. *There is a unique solution of the restricted approximation problem.*

Proof. Let $u_1 \neq u_2$ be solutions. Then

$$\bar{u} = \tfrac{1}{2}u_1 + \tfrac{1}{2}u_2$$

is a solution, too, which cannot be bang–bang, in contradiction to Theorem 3 if both u_1 and u_2 have this property.

References

1. F. R. Gantmacher and M. G. Krein, "Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme," Akademie-Verlag, Berlin, 1960.
2. K. Glashoff and N. Weck, Boundary control of parabolic differential equations in arbitrary dimensions: Supremum–norm problems. *SIAM J. Control Optimization* **14**, No. 4 (1976), 662–681.
3. R. B. Holmes, "A Course on Optimization and Best approximation," Lecture Notes in Mathematics No. 257, Springer-Verlag, Berlin/Heidelberg/New York, 1972.
4. A. Karafiat, The problem of the number of switches in parabolic equations with control. *Ann. Pol. Math.* **34** (1977), 289–316.
5. S. Karlin, "Total Positivity," Vol. I, Stanford Univ. Press, Stanford, California, 1968.